The Applications of Cotangent Bundle in Dynamics

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Abstract
This study aims to recognize the cotangent bundle and its applications in Dynamics, torsion, curvature calculation, recognize of mathematical bundle and knowledge of differential geometry.
We followed an analytical induction mathematical method because it is a suitable for this research. We found the following some results: Differential geometry depends on external geometry which contains curvature and torsion , calculation by many method such as derivation method and applied cotangent bundle in dynamics.

Keywords : Applications , Cotangent Bundle , Dynamics

. Introduction :

Mathematics highlights visibly two dimensional:

Tendency towards abstract ideas which crystallizes the relations and that one which links material in connected group of ideas and principles. This research addresses differential geometry as life is really full of examples that describes life reality that we live.

Our objective is to study differential geometry and recognize cotangent bundle and its applications in dynamics .

1. **Mathematics Bundles :**

i. **Vector Bundles:**

Let $M \subset R^*$ be an m-dimensional smooth manifold a smooth vector bundle (over M of rank n) is a smooth sub manifold $E \subset M \times R^L$ such that for every $P \in M$ the set.

$$Ep := \{v \in R^L | (p, v) \in E\}$$

Is an n-dimensional linear subspace of R^L (called the fiber of E over p) . If $E \subset M \times R^L$ is a vector bundle then a (smooth) section of E is smooth map s : $M \to R^L$ such that $s(p) \in Ep$ for every $p \in M$. A vector bundle $E \subset M \times R^L$ is equipped with a smooth map

$$\pi : E \ \to M \qquad (1)$$

Defined by $\pi(p, v) := p$ called the projection A section $s : M \to R^L$ of E determines a smooth map $\sigma : M \to E$ which sends the point $p \in M$ to the pair $(p, s(p) \in E$ this map satisfies. $\pi 0 \sigma = id$ (2)

It is sometimes convenient to abuse eliminate the distinction between s andσ. Thus we will sometimes use the same letter s for the map from M to E. [2].

Definition (2.1):

Any smooth map s : $B \to E$ such that $\pi os = idB$ is called a section of E if S is only defined over neighbourhood in B it is called a local section [9].

Definition (2.2)

A smooth section of a vector bundle (E,Mπ) is a map S:M → E so that πos = id M that is $s(p) \in EP$ for all $p \in M$ S is called smooth section if it is smooth as a map from M to E denote $\Gamma(E) = \{smooth\ section\ of\ (E, M\ \pi)\}$ [20]

Corollary (2.3)

Let $M \subset R^k$ be a smooth m- manifold. Then TM is vector bundle .over M and hence is a smooth 2m-manifold in $R^k \times R^k$.

Proof:

Let $\emptyset : u \ \to \Omega$ be a coordinate chart on an m-open set U⊂M with values in an open sunset $\Omega \subset R^m$. Denote its inverse by $\psi : \emptyset^{-1} : \Omega \ \to M$. The linear map $d\psi(x) : R^m \to R^k$ is injective and its image is $T_{k \times m} \psi(x) M$ for every $x \ \in \Omega$

Hence the map D : $U \ \to \ R^{k \times m}$ defined by

$$D(p) := \ d\psi\ (\phi(p) \in R^{k \times m} \qquad (3)$$

Is smooth and for every $p \in U$, the linear map $D(p) R^m \to R^k$ is injective and its image is.Thus the function $\pi^{TM} : M \to R^{K \times K}\ defined$ by $E_p : T_p M$ is given by.

$$\pi^{TM}(p) = \ D(p) D(p)^T D(p)^{-1} D(p)^T \text{ for } p \in U$$

Hence the restriction of π to u is smooth since M can be coverd by coordinate charts is follows that π^{TM}

Is smooth and hence by theorem TM is a smooth vector bundle [5].

ii. Fiber Bundle:

Fiber bundles are special types of manifold which are locally product of a base manifold B with a fiber manifold F.

To begin we define the cotangent bundle (E) over abase manifold (M) a manifold (e) with smooth projection map $\pi : E \ \to B$ onto a manifold (B).

The inverse image π_x^{-1} of a point $x \in B$ is called the fiber Fx above the point x we define $\{E, \pi, B, F, R\}$ consisting of

a. A manifold E projection map π, base B, fibre F together with structural group G of diffeomorphism of F acting on the left

b. An atlas of charts i.e a covering of B by open set U_i where i indexes the sets and maps ϕ_i called local trivializations such that.

$$\phi: \pi^{-1}(U_i) \rightarrow U_i \times F \qquad (4)$$

Where :

$$\phi_i(p): \{\pi(p), g_i(p)\}\ p \in \pi^{-1}(U_i)$$

and

$$g_i: \pi^{-1}(U_i) \rightarrow F \qquad (5)$$

Moreover if we define the restriction

$$g_i(x) = g_i|Fx$$

Then :

$g_i(x) = Fx \rightarrow F$ is a left action of G on F

c. Compatibility conditions such that $\forall\, v_i, v_j = v_i \cap v_j \neq \phi$ and if we define transition function by :

$$g_{ij}(x) = g_i(x) o g_i^{-1} = F \rightarrow F \qquad (6)$$

Then $\forall\, U_i, U_j, U_k : U_i \cup U_j \cup U_k \neq \phi$

$$g_{ij}(x) g_{ik}(x) = g_{ik}(x) \forall x \in U_i \cup U_j \cup U_k$$ [2]

iii. **Concept of Fiber Bundle**

The concept of a fiber bundle actually is comprised of two manifolds B and M and a surjective map.

$$\pi: B \rightarrow M \qquad (7)$$

(called the canonical projection) .All the preimages $F_x \equiv \pi^{-1}(x)$ are required to be diffeomorphic to a common manifold F and in addition each F_x is to be a submanifold in B (so it is to be nicely placed in B) .The last item of the definition is the requirement of local product structure : there exists a covering $\mathcal{O}_a$ of the base M and a system of diffeomorphisms

$$\psi_a = \pi^{-1}(\mathcal{O}\text{a}) \rightarrow \mathcal{O}_a \times F$$

(the map ψ_a is called a local trivialization such that $\pi 0 \psi_a = \pi$ [10]

Definition (2.4)

The quadruple (E, B, F, π) is called a smooth fiber bundle (or smooth fib ration of around each point of B there exists an open neighborhood U and diffeomorphism

$$\phi U = U \times F \rightarrow \pi^{-1}(U), \pi(\phi u(x,y) = x$$

For all $x \in U$ and y $\in F$

We call E the total space, B the base space , F the fiber space and π the projection map [7]

iv. Frame Bundle $\pi: LM \rightarrow M$

Define a map $\pi : LM \rightarrow M$, $e(x) \longmapsto x$

i .e we assign to a frame $e(x)$ in (the tangent space of) a point x just x itself .check that.

i. It is a smooth map a coordinate presentation (398)

$$\pi = \left(x^i , y^a_b\right) \longmapsto x^i \qquad (8)$$

ii. For arbitrary x the preimage $\pi^{-1}(x)$ is diffeomorhic to GL (n, R) so that for any two points $x, \acute{x} \in M, \pi^{-1}(x)$ and $\pi^{-1}(\acute{x})$ are diffeomorphic to each other) [10].

Example (2.5)

Frame bundle and associated tangent bundle it is possible to regard the frame bundle for R^n considered as an affine space, as the affine group F$(R^n) = A(n) = R^n \bowtie GL\,(n, R)$ and the base space as the coset R^n = GL $(n, R)/A(n)$ .The projection map π assigns each element of $A(n)$ to its coset with respect to GL(n, R) .To make this more concert recall that $E = A(n) = [s, x]$ may be given a matrix representation.

$$\begin{pmatrix} s & x \\ 0 & 1 \end{pmatrix}$$

With S∈GL (n, R) and x column n a vector acts on R^n considered as the column vector.

$$\begin{pmatrix} x \\ 1 \end{pmatrix}$$

The projection map π maps to

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

Which may be affected by left multiplication by

$$\begin{pmatrix} \bar{s} & 0 \\ 0 & 1 \end{pmatrix} \qquad [2]$$

v. The Orthonormal Frame Bundle :

The orthonormal frame bundle of M is the set O (M) :=$\{(p, e) \in R^n \times R^{n\times \mathrm{m}}|_p \in M$ ime $=$ TpM, $eTe =$ mxm

If we denote by

$$ei := e(0, \ldots, 0,1,0, \ldots., 01)$$

The basis 0 T_PM inducted by the isomorphism

$e: R^m \rightarrow T_p \;\; M$ then we have

$e^T e = 1 \Leftrightarrow \langle ei, ej \rangle = \sigma ij \Leftrightarrow ei, \dots, em$ is an orthonormal basis.Thus O (M) is the bundle of orthonormal frames of the tangent spaces $T_p M$ or the bundle of orthogonal Isomorphisms $e: R^m \rightarrow T_p M$.it is a principal bundle over M with structure group $\mathcal{O}(m)$ [5]

Example (2.5)

Observe that if M is homeomorphism to R^n , then we expect TM to be homeomorphic to R^{2n} , we just take a logbal coordinate chart $\phi = x$ on M get at each $p \in M$ a basis of vector $\left.\frac{\partial}{\partial x^k}\right|_p$ and then say that ourc coordinates on TM are given by the following rule : if $v \in T_p M$ expressed as $v = \sum_{i=1}^{n} a^i \left.\frac{\partial}{\partial x^i}\right|_p$ $where\ x(p) = (q^1, \dots, q^n)$ then the coordinate chart Φ on TM will be given by

$$\Phi(v) = (q^1, \dots, q^n, a^1, \dots, a^n)$$

Since Φ is a globally – defined chart , every P and every $v \in T_p M$ has a unique representation in this way , and conversely the coordinates $(q^1, \dots, q^n, a^1, \dots, a^n)$ we set $p = x^{-1}\,(q^1, \dots, q^n)$ and $v \in T_p M$ to b $v = \sum_i a^i \left.\frac{\partial}{\partial x_i}\right|_p$ [12]

vi. The Tangent Bundle :

Definition (2.6) :

The tangent bundle TM of a manifold M is (as a set) the (disjoint) union of all tangent spaces to M at all points $p \in M$.

$$TM = \left\{ (p, X_p) \in M \times \bigcup_{p \in M}^{T_p M} = X_{p \in} T_p M \right\}$$

The bundle projection $\pi = TM \longmapsto M$ is defined by $\pi(p, xp) = p$ .The fiber over $p \in M$ is the preimage $\pi^{-1}(p) = \{p\} \times T_p M$

A section of TM or tangent vector field is a map X = $M \;\; \longmapsto TM$ that satisfies $\pi\ oX = id\ M$ [22]

vii. The Tangent Bundle (TM) of the Tangent Bundle

The tangent bundle TM is a smooth 2m – dimensional manifold in $R^n \times R^n$ the tangent space of TM at a point $(p, v) \in TM$ can be expressed in terms of the second fundamental form as.

$$T_p M \;\; TM = \{(p\hat{\ }, v\hat{\ }) \in R^n \times R^n|_{p^\wedge} \in T_p M \qquad (9)$$

$$\mathbb{1} - \pi(p))\hat{v} = hp\ (\hat{p}, v)$$

By the gauss Weingarten formula the derivative of a curve $t \longmapsto (\gamma(t), X(t)$ in TM satisfies

$(\mathbb{1} - \pi\big(\gamma(t)\big)\bar{X}(t) = h_{\gamma(t)}(\bar{\gamma}(t), X(t))$ for every t.

This proves the inclusion in (3-8) Equality follows from the fact that both sides of the equation are 2m-dimensional linear subspace of $R^n \times R^n$ Now it follows from (3-8) that the formula .

$\gamma(p, v) := (v, hp(v, v) \in T(\text{p}, \text{v})TM$ for $p \in M$ and $v \in T_p M$ defines a vector field on TM [5]

We have defined the tangent bundle of a manifold as the disjoint union of the tangent space $TM = U_{p \in M} T_p M$ [4]

3.Curvature and Torsion

i. Curvature :

Definition (3.1)

Let C be a smooth curve with position vector $\vec{r}(s)$ where S is the length parameter the curvature k of C is defined to be

$$k = \left\| \frac{\overrightarrow{dT}}{ds} \right\|$$

Where $\vec{r}$ is the unit tangent vector [15]

Definition (3.2)

The magnitude of $\bar{T}(s)$ is called curvature k (at the point given by the vector $R(s)$

$$k = k(s) = |\bar{T}(s)| \qquad [18]$$

Definition (3.3)

The vector $N(s)$ is called the principal normal vector with this definition we have

$$\bar{T}(s) = k(s)N(s) \qquad [18]$$

Theorem (3.4)

Let C be a smooth curve with position vector $\vec{r}\,(t)$ where t is any parameter. Then the following formulas can be used to compute is

$$k = \frac{\left\| d \underset{T}{\overset{\rightarrow}{}}` \right\|}{\left\| \overrightarrow{r`}(t) \right\|} \qquad (10)$$

$$k = \frac{\left\| \overrightarrow{r`}(t) \times \overrightarrow{r``}(t) \right\|}{\left\| \overrightarrow{r``(t)} \right\|^3} \qquad (11)$$

Proof

We prove each formula separately

1- Proof $k = \frac{\left\| \overrightarrow{dT}`(t) \right\|}{\left\| \overrightarrow{r}`(t) \right\|}$ *using* the chain , we have $\frac{d\vec{T}}{dt} = \frac{d\vec{T}}{ds}\frac{ds}{dt} = \| \vec{r}\,'(t) \|$ *from* formula. $\frac{d\vec{T}}{ds} = \frac{\frac{d\vec{T}}{dt}}{\left\| \overrightarrow{r}`(t) \right\|}$

$$= \frac{\vec{T}`(t)}{\|\vec{r}`(t)\|}$$

2- proof of k = $\frac{\left\| \vec{r}\,' \times \vec{r}\,'' \right\|}{\| \vec{r}\,'(t) \|^3}$

We express $\overrightarrow{r`}(t)$ and $\overrightarrow{r``}(t)$ in terms of T : then compute their cross product.

Computation of $\overrightarrow{r`}$ since $\vec{T} = \frac{\overrightarrow{r`}}{\left|\overrightarrow{r`}\right|}$ and $\frac{ds}{dt} = \left\|\overrightarrow{r`}\right\|$ we get that $\overrightarrow{r`} = \frac{ds}{dt}\vec{T}$

Computation of $\overrightarrow{r`}(t) \times \overrightarrow{r``}(t)$ from the two previous formulas and using the properties of cross products we see that.

$$\overrightarrow{r`} \times \overrightarrow{r``} = \frac{ds}{dt}\frac{d^2s}{dt^2}\left(\vec{T} \times \vec{T}\right) + \left(\frac{ds}{dt}\right)^2 T \times \overrightarrow{T`}$$

$$\left\|\overrightarrow{r`} \times \overrightarrow{r``}\right\| = \left(\frac{ds}{dt}\right)^2 \left\|\vec{T} \times \overrightarrow{T`}\right\|$$

$$= \left(\frac{ds}{dt}\right)^2 \left\|\vec{T} \times \overrightarrow{T`}\right\| \sin\theta$$

We know that $\vec{T} \perp \overrightarrow{T`}$

Thus $\left|\overrightarrow{r`} \times \overrightarrow{r``}\right| = \left(\frac{ds}{dt}\right)^2 \left|\overrightarrow{T`}\right\| = \left|\overrightarrow{r`}\right\|^2 \left\|\overrightarrow{}\right\|$

There fore $\left\|\overrightarrow{T`}\right\| = \frac{\left|\overrightarrow{r`} \times \overrightarrow{r``}\right\|}{\left|\overrightarrow{r`}\right|^2}$

$K = \frac{\|\vec{T}\,'\|}{\|\vec{r}\,'(t)\|} = \frac{\|\vec{r}\,' \times \vec{r}\,''\|}{\|\vec{r}\,'(t)\|^3}$ [15]

iii. The Normal Curvature and Geodesic Curvature

Definition (3.5)

The scalars $kn(t0)$ and k_g (t_0) are called the normal curvature and the geodesic curvature of $\propto$ at the point $p = \propto (to)$ note that, from above we have

$$kn\,(t) = \acute{\acute{\propto}}\,(t)n(t) \qquad kg(t) = \acute{\acute{\propto}}\,(t).\,(n(t) \times \acute{\propto}\,(t)\ [13] \qquad (12)$$

iv. Curvature Computation :

Consider a parameterized curve $r(t) =$

$(x(t), y(t))$ and assume that $\langle t, n \rangle$ forms a right – hand basis

The curvature $k(t)$ is given by

$$k(t) = \frac{\acute{x}\grave{\grave{y}} - \acute{\acute{x}}\acute{y}}{(\acute{x}^2 + \grave{y}^2)^{\frac{3}{2}}} \qquad [19]$$

Example (3.6)

An ellipse is described parametrically by the equations .

$$x = 2\cos t\,, \qquad y = \sin t\,; \quad 0 \le t \le 2\pi$$

Solution :

First found $\acute{x} = -2\sin t\,, \acute{y} = \cos t$

$$\acute{\acute{x}} = -2\cos t\ , \acute{\acute{y}} = -\sin t$$

$$\because k(t) = \frac{\acute{x}\acute{\acute{y}} - \acute{\acute{x}}\bar{y}}{(\acute{x}^2 + \bar{y}^2)^{\frac{3}{2}}}$$

$$k(t) = \left| \frac{2\sin^2 t + 2\cos^2 \mathrm{t}}{4\sin^2 t + \cos^2 \mathrm{t})^{\frac{3}{2}}} \right.$$

$$= \frac{2}{[\,1 + 3\sin^2 \mathrm{t}]^{\frac{3}{2}}} \qquad [15]$$

Example (3.7):

Compute the normal and geodesic curvature of the circle $\sigma(t) = \cos t\,, \sin t\,, 1)$ on the elliptic parabolic $\propto (u, v) = (u, v, u^2 + v^2)$

Solution :

First we note that $sin^2 t + cos^2 t = 1$ so the curve $\propto (t)$ is contained in the surface $\sigma\ (u, v) = (u, v, u^2 + v^2)$ we need to compute $\bar{\propto}\,(t), n(t)$ and $n(t) \times\ \bar{\alpha}(t)$ in fact $\bar{\propto}\,(t) = (-\sin t, \cos t, 0)$

$\|\bar{\propto}(t)\| = 1$ so $\propto$ is the arc – length parameterization .To find $n(t)$ we note that $n(t)$ is the restriction of n to the curve $\propto$. So we first calculate n since $\sigma u = (1,0,2u)$ $\partial_v = (0,1,2v)$

$$\sigma_u \times \sigma_v = (-2u - 2v, 1), \|\sigma_n \times \sigma_v\| = \sqrt{1 + 4u^2 + 4v^2}$$

$$n = \left(\frac{-2u}{\sqrt{1 + 4u^2 + \ 4u^2}}, \frac{-2v}{\sqrt{1 + 4u^2 + 4v^2}}, \frac{1}{\sqrt{1 + 4u^2 + 4v^2}}\right)$$

We need to write $\propto(t) = \sigma\big(u(t), v(t)\big)$ means that

$(cost\,, \sin t, 1) = < u(t), v(t), u^2(t) + \ v^2(t)$.

This implies that $u(t) = cost\,, v(t) = \sin t$

The restriction of n to the curve $\propto$ is taking $\grave{u}(t) = \cos t\; v(t) = \sin t$

$$n(t) = \ n(\propto(t) = \left(\frac{-2}{5}\cos t\,, -\frac{2}{\sqrt{5}}\sin t\,, \frac{1}{\sqrt{5}}\right.$$

Finally $n(t) \times \acute{\propto}(t) = \left(-\frac{1}{\sqrt{5}}\cos t\,, \frac{-1}{\sqrt{5}}\sin t, \frac{1}{\sqrt{5}}\right)$

To find the normal curvature kn we note that in $kn = \acute{\acute{\propto}}(t).n(t)\; since\; \acute{\acute{\propto}}(t) = (-cost\,, -sint\,,0)$ we have .

$kn(t) = \acute{\acute{\propto}}(t).n(t) = \ \frac{2}{\sqrt{5}}]$

Similarly

$kg^{(t)} = \acute{\acute{\propto}}(t).n(t) \times \acute{\propto}(t) = \ \frac{1}{\sqrt{5}}$ [9]

Example (3.8)

Find the curvature of circular helix earlier we found that the parameterization of the circular helix with respect to arc-length was

$$\vec{r}(s) = (\cos\frac{s}{\sqrt{2}}\,, sin\,\frac{s}{\sqrt{2}}\,, \frac{s}{\sqrt{2}})$$

Solution

As before we need to compute $\vec{T}(s)$ which can be obtained from $\vec{\acute{r}}(s)$

$\vec{\acute{r}}(s) = (-\,\frac{1}{\sqrt{2}}\,sin\,\frac{s}{\sqrt{2}}\,, \frac{1}{\sqrt{2}}\cos\frac{s}{\sqrt{2}}\,, \frac{1}{\sqrt{2}})$

Thus :

$$\vec{T}(s) = (-\frac{1}{\sqrt{2}} sin\ \frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}} \cos\frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}})$$

Before

$$\frac{d\vec{T}}{ds} = \frac{1}{\sqrt{2}}\ (\frac{-1}{\sqrt{2}} \cos\frac{s}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \sin\frac{s}{\sqrt{2}}, 0\ )$$

It follows that :

$$k = \left\|\frac{d\vec{T}}{ds}\right\| = \frac{1}{2} \qquad [15]$$

v. The Gauss Curvature and Mean Curvature:

Definition (3.8) :

Let M be a surface and p∈ M . Let k_1 ,k_2 be the principal curvatures of M at p.Then $k = k_1 k_2$ is called the Gussian curvature of M and

$$\mathrm{H} = \frac{k_1 + k_2}{2} \qquad (13)$$

Is called the mean curvature of M . [9]

Example (3.9):

Consider a surface of revolution

$\delta(u, v) = (\phi(u) \cos v\ , \phi(\ u\ sin\ v, \psi(u)$

Where $\grave{Q}^2 + \grave{\psi}^2 = 1$. As we calculated $E = 1 \quad F = 0$

$$Q = \phi^2(u) \text{ and } e = \acute{\phi}(u)\acute{\acute{\psi}}(u) - \acute{\acute{\phi}}(u)\ \acute{\psi}\ (u) \qquad (14)$$

$f = 0 \ \ g = \phi(u)\acute{\psi}(u)$ Hence

$$F_1 = \begin{pmatrix} 1 & 0 \\ 0 & \phi^2(u) \end{pmatrix} = F_{11} = \begin{pmatrix} \acute{\phi}\ (u)\acute{\acute{\psi}}(u) - \acute{\acute{\phi}}(u)\acute{\psi}(u) & 0 \\ 0 & \phi(u)\acute{\psi}(u) \end{pmatrix}$$

$$A = F_1^{-1}\ F_{11} = \begin{pmatrix} \acute{\phi}\ (u)\acute{\acute{\psi}}(u) - \acute{\acute{\phi}}(u)\acute{\psi}(u) & 0 \\ 0 & \grave{\psi}(u)/\phi(u) \end{pmatrix}$$

So its Gauss curvature is using $\grave{\phi}^2 + \grave{\psi}^2 = 1$

$$K = \det(A) = \frac{\phi'(u)\psi''(u) - \phi''(u)\,\psi'(u)\psi'(u)}{\phi(u)}$$

$$= \frac{-\phi''(u)}{\phi(u)}$$

And its mean curvature is

$$H = \frac{1}{2}\, trace\,(A)$$

$$= \frac{1}{2}(\phi'(u)\psi''(u) - \phi''(u)\psi'(u) + \frac{\psi'(u)}{\phi(u)} \qquad [9]$$

Example (3.10) :

Let us compute the Gaussian curvature of the metric g= $dx^2 + 2cos$ w $dx\,dy + dy^2$ where $w = w(x, y)$ is some function .Represent this metric in the form

$$g = (dx + \cos w\, dy)^2 + (\sin w\, dy)^2$$

we may set $u_1 = dx + \cos w\, dy$, $u_2 = \sin w\, dy$ and get

$\delta = u_1 \wedge u_2 = \sin w\, dx \wedge dy$

Differentiating the basic forms we get

$$du_1 = -\sin ww\, x\, dn \wedge dy \quad du_2 = \cos ww\, x\, dx \wedge dy$$

Therefore

$$\propto_1 = -w_x \quad \propto_2 = \cot w \;\; \theta = \propto_1 u_1 + \propto_2 u_2 = -w_x dx$$

Differentiating we get

$$d\theta = wxy\, dx \wedge dy \qquad k = \frac{-wxy}{\sin w}$$

The metric is flat $(k = 0)$ if $w$xy $= 0$ let us determine the Euclidean coordinates for the case $w = x + y$ in this case the form θ is exact $\theta = -d\psi$ with $\psi(x, y) = x$ rotating the frame u_1, u_2 by the angle ψ we get.

$$u'_1 = \cos\psi\, u_1 + \sin\psi\, u_2 = \cos x\, dx + (cosx \cos(x, y)$$

$$+ \sin x(sinx + y)dy = \cos x\, dx + \text{cosy}\, dy = d\,(\sin x + \sin y)$$

$$\bar{u}_2 = \sin\psi\, u_1 - \cos\psi\ u_2 = \sin x \cos(x+y)$$

$$-\cos x \sin(x+y)dy = \sin x\, dx - \sin y\ dy$$

$$= d(-\cos x + \cos y)$$

The desired Euclidean coordinates are $X = \sin x + \sin y$ and $y = -\cos x + \cos y$ [9]

vi. Gaussian Curvature :

Definition (3.11)

The Gaussian curvature of the hyper surface M is the real valued function

$K: M \to R$ defined by

$$k(p) := \det(dv(p): T_p\ M \to T_p M \qquad (15)$$

For $P \in M$ replacing v by $-v$ has the effect of replacing k by $(-1)^m k$; so K is independent of the choice of the Gauss map when m is even [5]

Definition (3.12)

At a point P an a surface S the Gauss curvature at P is the limit

$$k = \lim_{\Delta A \to 0} \frac{\Delta\theta}{\Delta A} \qquad (16)$$

Where ΔA is the area of some region the surface containing P and $\Delta\theta$ is the total curvature of that region [1]

viii. Torsion :

Definition (3.13) :

Let $\propto: I \to R^3$ be a curve parameterized by arc length s. The torsion of $\propto$ at s is defined by:

$$\tau_{(s)} = \acute{N}(s).B(s) \qquad (17)$$

Now we can express $\acute{N}(s)$ as

$$\acute{N}(s) = -k(s)T(s) + \tau(s)B(s) \quad [12] \qquad (18)$$

Curvature vs, Torsion(3.15)

The curvature indicates how much the normal changes in the direction tangent to the curve.

The torsion indicates how much the normal changes in the direction orthogonal to the osculating plane of the curve.

The curvature is always positive, the torsion can be negative.

Both properties do not doped on the choice of parameterization.

What is $\acute{B}(s)$as a combination of N,T,B?

$$\text{We know } B(s).B(s) = 1 \qquad (19)$$

Form the lemma → $\acute{B}(s).B(s) = 0$

We know :

$$B(s).T(s) = 0 \quad B(s).N(s) = 0 \qquad (20)$$

From the lemma →

$$\acute{B}(s).T(s) = -B(s).T(s) \qquad (21)$$

$$-B(s).k(s)N(s) = 0$$

From the lemma

$$\acute{B}(s).N(s) = -B(s).N(s) \qquad (22)$$

Now

We can express $\acute{B}(s)$ as :

$$\acute{B}(s) = -\tau(s)N(s)$$

Proposition (3.16) :

(a) if $k(s) = 0$ for all $s \in I$ then $\propto (s)$ is part of a straight line

(b) If $T(s) = 0$ for all $S \in I$ then $\propto (s)$ apalnar curve that is it lies inside some plane in R^3

Proof :

(a) If $k(s) = 0$ for all $S \in I$ then $\propto (s)$ is part of straight line.

If $k(s) = 0$ the $\frac{dT}{ds} = 0$ which implies that $T = a$ for some constant vectora with $|a| = 1$ then since $T = \acute{\propto} (s)$ intergrading a gain , we obtain $\propto (s) = as + b$

(b) If $T(s) = 0$ for all $s \in I$ then $\propto (s)$ is a planer curve , that is it lies inside some plane in R^3
If $T(s) = 0$ then we have.

$$\acute{B} = 0$$

This implies that B(s) is a constant vector which we just denote as B, we claim that $\propto (s)$ lies in a plane of which B is normal vector In particular proving the claim will finish the proof of the proposition

To prove the claim we need to see that $B. \propto (s)$ to obtain .

$$\acute{B}. \propto (s) + B. \acute{\propto} (s) = B. \acute{\propto} (s) = B.T(s) = 0 \qquad (23)$$

Since $B = T(s) \wedge N(s)$then $(B. \acute{\propto} (s) = 0$ implied $B. \propto (s) is$ a constant C say so that $\propto (s)$ this liens in plane $r.B = c$ as claimed. [16]

Definition(3.17) :

The vector $N(s)$ is called the principal normal vector with this definition we have

$$\acute{T}(s) = k(s) \ N(s)$$

$$B(s) = T(s) \times N(s) \qquad (24)$$

$\sin ce$ $B(s)$ is a unit vector [12]

Example (3.18)

Find T, N, B, k ,and T for $r(t) = (6 \sin 2t) \ i + 6 \cos 2t \ j + 5 \ tk$

First

$$\acute{r}(y) = (12 \cos 2 \ t) \ i - (12 \sin(2t) j + 5k$$

$$\acute{\acute{r}}(y) = (-24 \sin(2t) i - (24 \cos 2t) j$$

This means we have $T(t) = \frac{\vec{r}(t)}{|\vec{r}(t)|}$

$$= \frac{12 \cos 2 \ ti - 12 \sin(2t) j}{\sqrt{144 \ cos^2(2t) + 144 sin^2(2t) + 25}}$$

$$= \frac{12 \cos(2t) i - 12 \sin(2t) j}{\sqrt{169}}$$

$$T(t) \;=\; \frac{12}{13}\cos 2ti - \frac{12}{13}\sin(2t)j + \frac{5}{13}\;k$$

$$\acute{T}(t) \;=\; \frac{-24}{13}\sin(2t)i\;\frac{-24}{13}\cos(2t)j$$

This has magnitude $\frac{24}{13}$, so we divide by this to get the unit normal vector N.

$$N(t) = \,-(\sin(2t)i - (\cos(2t)\,j$$

The unit binormal vector is the cross product of the unit tangent and normal vectors.

$$B(t) = \,T(t) \times N(t) = \begin{vmatrix} i & j & k \\ \frac{12}{13}\cos(2t) & -\frac{12}{12}\sin(2t) & \frac{5}{13} \\ -\sin(2t) & -\cos(2t) & 0 \end{vmatrix}$$

We can find the curvature b

$$k \;=\; \frac{\|\bar{r} \times \acute{r}\|}{|\acute{r}|^2} \;=\; \frac{24}{169}$$

The torsion is then given by

$$\tau = \frac{\begin{vmatrix} \acute{x} & \acute{y} & \acute{z} \\ \acute{\acute{x}} & \acute{\acute{y}} & \acute{\acute{z}} \\ \acute{\acute{\acute{x}}} & \acute{\acute{\acute{y}}} & \acute{\acute{\acute{z}}} \end{vmatrix}}{\left|\acute{r} \times \acute{\acute{r}}\right|^2} \;=\; \begin{vmatrix} 12\cos 2t & -12\sin(2t) & 5 \\ -\;24\sin(2t) & -24\cos(2t) & 0 \\ -48\cos(2t) & 48\cos(2t) & 0 \end{vmatrix}$$

$$5\;\;(-24\;\times 48\;sin^2(2t) - 24\;\times 48\;cos^2(2t)$$

$$\tau = \,-5\;\frac{(1152)}{3122} = \,-\frac{5760}{3122} \qquad\qquad [12]$$

4.The Cotangent Bundle and its some Applications

i. **The Cotangent Bundle :**

Let M $_n$ be an n-dimensional differentiable manifold of class C^∞ and $T^*\;(M_n)$ the cotangent bundle over M_n . If x^i are local coordinates in neighborhood U of a point $x \in M_n$, then a covector p at x which is , an element of T^*M_n , is expressible in the from $(x^i, p^i)$where p_i are components of p with repect to the normal frame ∂_i we many consider

$(x^i, p_i) = (x^i, \overline{x^{\bar{\imath}}}) = (x^j), i = 1, .. n;\ \bar{\imath} = n + 1, \dots, 2n\ , J = 1 \dots 2n$ as local coordinate in a neighborhood $\pi^{-1}(U)$ (π is natural projection $T^*(M_n)$ on to M_n.

Let now M_n be a Riemannian with nondegenarate metric whose components a coordinate neighborhood U are gij and denote by Γ_{ji}^h the christoffe symbols formed with gji [12]

Definition (4.1)

Let us define the cotangent bundle of a manifold M to be the set

$$T^*M: = \bigcup_{P \in M} T_p^*\ M$$

and define the map $\pi: = \ \pi M = U\ T\text{p}^*M \to M$

to be the obvious projection $p \in M$

taking element in space to the corresponding p [4]

Theorem (4.2)

Regular cotangent reduction at zero .Let G act freely and properly by cotangent lifts on T^*Q with momentum m J, Denote $\pi_G: Q\ \to Q/G\ i = J^{-1}(0) \to T^*Q$

And $\pi_0: J^{-1}(0) \to J^{-1}(0)\lambda(G)$ the natural quatient maps and inclusion consider.

$$\psi = \ J^{-1}(0) \longrightarrow T^*(Q/G) \qquad (25)$$

Defined by $(\psi(z), T_q\pi_G(v)) = (z, v)$ for ever z$\in T_q^*Q$ and $v \in T_qQ$. The map ψ is a $G - M$ variant surjective submersion that induce a symplectomorphism

$$\psi = \ J^{-1}(0)/G \to T^*(Q/G) \qquad (26)$$

Where $J^{-1}(0)/G$ is endowed with the reduced symplectic from w_0 that is the one satisfying $\pi_0^* w_0 = \ i^* w_Q$ [11]

ii. **The Cotangent Bundle T_p*M and Forms :**

Definition (4.3):

Suppose $P \in M$is any point, and let F be a germ of a function at p. That is there is some open $U \ni p$ and a smooth function $f: V \to R$. We define $df_p = \ T_pM \to R$ to be the operation $df|_p(v) = v(f)$.

The operator $df|_p$ is linear and hence $df|_p \in T_p^* M$, the dual space of $T_p M$. it is called the differential of f at p .In general , elements of $T_p^* M$ are usually called cotangent vectors or 1- forms. [5] **Example (4.4):**

Suppose $f: c \to R$ is given by $f(z) = lm(e^z)$. Then is coordinate we have $f(x,y) = e^x$ siny writting $v = a \left.\frac{d}{dx}\right|_0 + b \left.\frac{d}{dy}\right|_0$ we have

$$df|_0(v) = v(f) = a\frac{d}{dx}(e^x smy)(0,0) + b\frac{d}{dy}(e^x smy)(0,0) = b$$

Hence the cotangent vectors satisfies

$df \ _0\left|\left(\left.\frac{d}{dx}\right|_0\right)\right) = 0$

$$df\left|\left(\left.\frac{d}{dy}\right|_0\right) = 1 \qquad [5]\right.$$

Theorem (4.5) :

Suppose that $di \geq c$ for $i = 1, \dots, k$ and

$$k\,(k-1) > \frac{8d^2(2d-5)}{c^2(d^2-1)}$$

Then the minimal resolution $y \to x$ has bis cotangent bundle

Proof:

The charn number of the minimal desingularization y of X are = $c_1^2 = d(d-4)^2$ $c_2 = d(d^2 - 4d + 6)$ the chern number of the orbifold × are $c_1^2(x) = c_1^2$ and

$$c_2(x) = c_2 - \left(d - \frac{1}{d}\right)\sum_{i<_j} d_i d_j \qquad (27)$$

Thus

$$s_2(y) \neq s_2(x) = 4d(5-2d) + \left(d - \frac{1}{d}\right)\left(\sum_{i<_j} d_i d_j\right.$$

$$> 4d(5-2d) + \frac{k(k-1)}{2}\left(d - \frac{1}{d}\right)c^2$$

As a corollary we obtain may examples of surfaces in p[3] with big cotangent bundle [21]

Example (4.6) :

Let $M = T^{*}N$ with canonical symplectic form $w = d\theta$ if we consider a smooth 1- form $\propto$ on N a smooth section $\propto$:N→ M of the cotangent bundle $\pi: M \to N$, then the sub manifold $\propto: N \to M$ is lagrangian if and only if $\propto$ is closed[7]

Theorem (4.7) :

Let $X,\ y\ \in\ \mathrm{T}_0^1\ (M_n)$. Then the inner product of the horizontal lifts H_x and H_y to $T^{*}\ (M_n)$ with the metric D_g is equal to the vertical of the inner product of x and y in M_n.

We have

$$D_g(v_w, v_\theta) = v(g(w,\theta) \forall\, w,\theta \in T_1^0\ (M_n)$$

$$D_g(v_w, c_x) = -g^{is}\ w_j p_L\big(\sigma_s X^L + \Gamma_{si}^{Lxi}$$

$$= -g^{is} w_j\ \big(L\ (\nabla x)\big)s$$

$$= -v(g\big(w, L(\nabla X)\big).$$

$$Dg\ \left(c_x, c_y\right) = g_{ji} x^j y^i + g^{ji} p_k p_L\big(\nabla_j X^k\big)(\nabla_i y^L)$$

$$= g_{ji} x^j\ y^i + g^{ji}\ (L\nabla X)\mathrm{j}\big(L(\nabla y)\big)i$$

$$= v\ (g(x,y) + v(g(L(\nabla X), L(\nabla y)) \forall X, y \in T_0^{-1}(M_0$$

$$\forall\, w\ \in T_1^0(M_n$$

Where $(L(\nabla X) is\ a\ 1 -$ from with local expression

$$L(\nabla X) = P_L \nabla_s X^L dx^s$$

We recall that any element $t \in T_r^0\ (T^{*}(M_n)$ is completely determined by its action on lifts of the type $X_1^c, X_2^c \ldots Cx_2$ where $X_i, i = 1, \ldots r$ are arbitrary vector fields in M_n [12]

iii. **The Cotangent Bundle of a Manifolds :**

Before we can introduce the Legendre transformation we need some basic facts about the structure of the cotangent bundle $T^{*}M$ of an n- dim differentiable manifold M , we suppose that M is the configuration space of some classical system

$$T^*M = \{(x,\lambda)|x \in M , \lambda \in T_x^*M \qquad (28)$$

= momentum phase space

= set of all kinematically possible states of motion

= a 2n – dim differentiable manifold

The projection map $\pi: T^*M \to M$ is defined by $\pi(x,\lambda) = x$ [7]

iv. The Cotangent Bundle of Question Variety.

Definition (4.9) :

The cotangent bundle of an orbit space X/G is the stratified symplectic space definition makes sense [13]

Example (4.10) :

Consider the action of G/L (n,R) the group of $n \times n$ inventible matrices or more properly, the group of invertible linear transformation of R^n to itselfType equation here..

$$\Phi\ A(q) = Aq$$

The group of induced canonical transformation of T^*R^n to itself is given

$$\Phi_{\dot{A}}^*(q,p) = (A^{-1}q, A^T q)$$

Which is readily verified notice that this reduces to the same transformation of q and p when A is orthogonal.[7]

Theorem (4.11) :

For any 1- from $\propto$ and vector filed X on N

$$\tilde{R}(\propto^v) = R(\propto)^v$$

$$\tilde{R}(\tilde{X}) = \tilde{R}(\tilde{X}) + (\mathcal{L}_x R)^v$$

Proof :

The proof consist essentially of repeated applications of the proceeding formula .To obtain the second result we use this formula with $\varepsilon = \tilde{X}$

$$d_\theta(\tilde{R}(\tilde{X}), B^v) = \mathcal{L}R^v(d\theta_N(\tilde{X}, B^v)) + d\theta_N((\mathcal{L}_x R)^v, B^v) + d\theta_N(\tilde{X}, R(B)^v)$$

$$= d\theta_N(\tilde{X}, R(B)^v)$$
$$= -\pi N^*(X, R(B))$$
$$= -\pi N^*(R(X), B)$$
$$= d\theta_N(\tilde{R}(X), (\tilde{B})$$

On the other hand

$$d\theta_N(\tilde{R}\widetilde{(X)}\tilde{Y}) = \mathcal{L}d\theta_N(\tilde{X}\tilde{Y})$$

$$+ d\theta_N((\mathcal{L}_x R)^v \tilde{y}) + d\theta_N(\tilde{X}, (\mathcal{L}y, R)^v)$$

$$= \mathcal{L}R^v[h[x, y] + d\theta_N(\mathcal{L}x, R)^v, \tilde{y}) - h\mathcal{L}y R(x)$$

$$= hR[X, Y] + d\theta_N(\mathcal{L}_x R)^v, \tilde{Y}) - h[Y, R(X) + hR[Y, X]$$

$$= d\theta_N((\mathcal{L}x R)^v, \tilde{y}) + h[R(x)y]$$

$$= d\theta_N\big((\mathcal{L}_x R), \tilde{y}\big) + d\theta_N(\tilde{R}(X), \tilde{y})$$

The second assertion of the theorem now follows the first assertion is easily verified by similar Considerations with $\mathcal{E} = \propto^r$ [8]

v. Application Discussion of the Dynamics :

In the last section we discussed the Hamiltonian dynamics of the reduced system ($T^*V //_{\mathcal{O}} H_0 = H_{cm}$) However, now we want to investigate.

The dynamical behavior an the reduced configuration space $V/G = \sum/W = C$

which is often also called the shape space of the system .This is interesting because the dynamics that take place on this space are those of the calogero- Moser dynamical system.

In the previous sections we have given two isomorphic descriptions of an open dense subset of the reduced system ($T^*V //_{\mathcal{O}} H_0 = H_{cm}$)

That is we are in the following situation

$$C_r \times (\sum \times \mathcal{O} //_0 M) \hookrightarrow T^*V // \mathcal{O}^G \hookleftarrow (\sum \times \mathcal{O} //_0 M) \times C_r$$

Where we have placed brackets to distinguish between reduced position and reduced momentum coordinates.

As a particular case consider the situation where $(L_0)^M$ is an element of the isotropy lattice such that.

$$(\mathcal{O} //_0 M)^m_{(L_0)} = (\mathcal{O}^M_{(L_0)} \cap A_{nnm}) / M$$

Is discrete .for Example, this was the case in the explicit approach of section $\psi.A$. since $W = W(\sum)$is a reflection group we conclude from the above that , in this case the dynamic on the shape space is given by a line in$(\sum)$that is reflection at all walls .Thus the scattering process is given by a trans formation of the type

$$(x_1, \dots, x_L) \longmapsto (x_L, \dots, x_1)$$

Where $L = dim(\sum)$

more generally, the dynamics are more complicated and we consider the coordinates of $(\mathcal{O} / /0M)(L0)m$ to be spin coordinates which keep the dynamics from hitting certain walls [13].

Results :

We found the following some results: Possibility of calculation curvatures by more than one method such as derivation method and we found that its easy to applied the cotangent bundle on dynamics .

Conclusion :

Finally we can say that dynamics is an important application of cotangent bundle .

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